Prop: Suppose that a function $F: \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(i) $0 \leq F(t) \leq 1$, for all $t \in \mathbb{R}$

(ii) F is a non-decreasing function, ie: if $t_1 \leq t_2$, then $F(t_1) \leq F(t_2)$

(iii). $\lim_{t \to \infty} F(t) = 1$ and $\lim_{t \to -\infty} F(t) = 0$

(iv) F is a right continuous function on $\mathbb{R}$

THEN: there is a prob space (Ω, ξ, P) and a r.v $X: \Omega \to \mathbb{R}$ such that $F_X = F$ on $\mathbb{R}$

$\square$

Observation: there are three kinds of salt, I, II, III.
We have the following informations.

	I	II	III
Price \$/kg	2	6	10
weight (kg)	4	8	12

Q: To mix all salt I, II, III together, what is the average of the price?

sol

~~Define $f: \Omega \to \{4, 8, 12\}$ Let $\Omega = \{I, II, III\}$~~
~~Define a function $X : \Omega \to \{4, 8, 12\}$ $\{2, 6, 10\}$~~
~~$X(I) = 2$, $X(II) = 6$, $X(III) = 10$~~

Let $\Omega = \{I, II, III\}$
Put $X : \Omega \to \{2, 6, 10\}$
$X(I) = 2$, $X(II) = 6$, $X(III) = 10$

Note
the average of the price =

$$\frac{2\times 4 + 6\times 8 + 10\times 12}{4+8+12}$$

$$= 2\times\frac{4}{4+8+12} + 6\times\frac{8}{4+8+12} + 10\times\frac{12}{4+8+12}$$

$$= 2\times P\{X=2\} + 6\times P\{X=6\} + 10\times P\{X=10\}$$

From now on, all rvs are assumed to be discrete.

Def: Let X be a rv with $\text{im } X = \{x_1, \ldots, x_N\}$ $(1 \le N \le \infty)$. The expectation of X (or mean of X) $(x_i \ne x_j,\ i \ne j)$, write $\mathbb{E}(X)$, is defined by

$$\mathbb{E}(X) \overset{⊛}{=} \sum_{i=1}^{N} x_i P\{X = x_i\}$$

provided ⊛ is absolute convergent, i.e.

$$\sum_{i=1}^{N} |x_i| P\{X = x_i\} < \infty$$

Remark ① the absolute convergence of ⊛ assures that the series ⊛ does not depend on the rearrangement.

②: if $A \in \Sigma$ then $E(I_A) = P(A)$.

Prop: Let X ~~,~~ Y be discrete rvs. We have

(i) $E(aX+b) = aE(X)+b$. a, b consts

(ii) $E(X+Y) = E(X)+E(Y)$

(iii) $E(g \circ X) = \sum g(x_i) P\{X = x_i\}$, $g: \mathbb{R} \to \mathbb{R}$ is a Borel function

pf (i): Let $\text{im}\, X = \{x_1, \ldots, x_N\}$ (assume $x_i \neq x_j$)

Then $\text{im}(aX+b) = \{ax_1+b, \ldots, ax_N+b\}$

Hence $E(aX+b) = \sum (ax_i+b) P\{aX+b = ax_i+b\}$

$= \sum (ax_i+b) P\{X = x_i\}$

$= a\sum x_i P\{X = x_i\} + b\sum P\{X = x_i\}$

$= aE(X) + bP(\Omega) = aE(X)+b$

(ii) Let $\text{im}\, Y = \{y_1, \ldots, y_M\}$ $(y_i \neq y_j)$

~~Then~~ Note $\text{im}(X+Y)$ ~~...~~

$\text{im}(X+Y) \subseteq \{x_i + y_j \mid i = 1, \ldots N, j = 1, \ldots M\}$

~~Hence $E(X+Y) = \sum_{\omega} X$~~

Then $E(X+Y) = \sum_{i,j} (x_i+y_j) P\{X+Y = x_i+y_j\}$

$= \sum_{i,j} (x_i+y_j) P\{X = x_i, Y = y_j\}$

$= \sum_{i,j} x_i P\{X = x_i, Y = y_j\} + \sum_{i,j} y_j P\{X = x_i, Y = y_j\}$

for all Borel subsets $A_1, \ldots, A_m)$

$$= \sum_i x_i \sum_j P\{X=x_i, Y=y_j\} + \sum_j y_j \sum_i P\{X=x_i, Y=y_j\}$$

$$= \sum_i x_i P\{X=x_i\} + \sum_j y_j P\{Y=y_j\}$$

$\square$

Def: Let X be a discrete rv. ~~Assume~~ The variance of X, write $\mathrm{var}(X)$, is defined by

$$\mathrm{var}(X) \equiv E[(X-\mu)^2] \quad \text{provided it exists}$$

where $\mu \equiv E(X)$.

Prop: (i) $\mathrm{var}(X) = E(X^2) - E(X)^2$

(ii) $\mathrm{var}(aX+b) = a^2 \mathrm{var}(X)$

(iii) ~~If X... are indep~~

if X, Y are indep rvs, then

$$\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y)$$

pf (i). $\operatorname{var}(X) \equiv E[(X-\mu)^2]$

$$= E(X^2 - 2\mu X + \mu^2)$$

$$= E(X^2) - 2\mu E(X) + \mu^2$$

$$= E(X^2) - \mu^2.$$

(ii). $\operatorname{var}(aX+b) = E[((aX+b)-\mu)^2]$

Note: $\mu_a \equiv E(aX+b) = a\mu + b$

Hence $\operatorname{Var}(aX+b) = E(aX+b-a\mu-b)^2$

$$= \cancel{E(a^2X^2)} = E(a(X-\mu))^2$$

$$= a^2 E(X-\mu)^2$$

$\square$

From now on, let $\mathcal{B}(\mathbb{R})$ be the σ-algebra on $\mathbb{R}$ generated by $\{(-\infty, a] \mid a \in \mathbb{R}\}$

In this case, each event $A \in \mathcal{B}(\mathbb{R})$ is called a Borel subset of $\mathbb{R}$

Prop: A function $X: \Omega \to \mathbb{R}$ is a r.v

iff $\{X \in A\} \in \xi \quad \forall A \in \mathcal{B}(\mathbb{R})$.

$\square$

Prop: Let $g : \mathbb{R} \to \mathbb{R}$ be a Borel function. Then (34)

$$E(g \circ X) = \sum g(x_i)\, P\{X = x_i\}$$

~~Ne~~ pf: let $\operatorname{im} g \circ X = \{y_1, \dots, y_M\}$, $\operatorname{im} X = \{x_1, \dots, x_N\}$

Then for each y_j, $\exists\, x_i \in X$ s.t. $g(x_i) = y_j$

Note $E(g \circ X) = \sum_j y_j\, P\{g \circ X = y_j\}$

$$= \sum_j \sum_{i : g(x_i) = y_j} g(x_i)\, P\{X = x_i\} \quad \Big(\because\ \{\omega \mid g \circ X(\omega) = y_j\}$$

$$= \{\omega \mid g(x_i) = y_j,\ X(\omega) = x_i\}$$

$$= \sum_i g(x_i)\, P\{X = x_i\} \qquad = \coprod_{i : g(x_i) = y_j} \{X = x_i\} \Big)$$

$\square$

Propn

~~Propn Let X, Y be rvs. Then~~

~~$\mathrm{var}(X+Y) = \mathrm{var}$~~

~~Def: Let X, Y be rvs. The covariance of between X and Y is defined by~~

~~$\mathrm{cov}(X, Y) = E(X - \mu_X)(Y - \mu_Y)$~~

Def: Let $X_1, \dots, X_N$ ($1 \le N \le \infty$) be a sequence of rvs. We say that $(X_1, \dots, X_N)$ are independent if for any finitely many subsequence $X_{i_1}, \dots, X_{i_m}$, and ~~for~~ for any real numbers $a_1, \dots, a_m$, $\{X_{i_1} \le a_1\}, \dots, \{X_{i_m} \le a_m\}$ are indep events.

Remark: $X_1, \dots, X_N$ are indep iff

$\{X_{i_1} \in A_1\}, \dots, \{X_{i_m} \in A_m\}$ are indep.

for all Borel subsets $A_1, \dots, A_m$)

Remark: Two events A, B are independent iff I_A, I_B are indep rvs

pf: "$\Rightarrow$":

Case: $a < 0$, $b < 0 \Rightarrow \{I_A \le a\} = \phi$, $\{I_B \le b\} = \phi$

$0 \le a < 1$, $b < 0 \Rightarrow \{I_A \le a\} = A^c$, $\{I_B \le b\} = \phi$

$0 \le a < 1$, $0 \le b < 1 \Rightarrow \{I_A \le a\} = A^c$, $\{I_B \le b\} = B^c$

$\vdots$

"$\Leftarrow$" Since $A^c = \{I_A \le \frac{1}{2}\}$ and $B^c = \{I_B \le \frac{1}{2}\}$

$\Rightarrow A^c, B^c$ are indep $\Rightarrow A, B$ are indep

Prop: Let X, Y be two indep rvs & Let $f, g: \mathbb{R} \to \mathbb{R}$ be two Borel functions.

Then $f \circ X$, $g \circ Y$ are indep rvs

and $E((f X) g(Y)) = E(f(X)) E(g(Y))$

pf: Note: $\{f \circ X \le a\} = \{X \in f^{-1}(-\infty, a)\}$

$\{g \circ Y \le b\} = \{Y \in g^{-1}(-\infty, b)\}$

$\therefore$ $f(X)$ and $g(Y)$ are indep.

Claim: $E(f(X) g(Y)) = E(f(X)) E(g(Y))$

3b

pf claim: Since $f(X)$, $g(Y)$ are indep,

need to show that if X, Y are indep, then

$$E(XY) = E(X)E(Y)$$

Let $Z \equiv XY$, and $\mathrm{im}\, Z \equiv \{z_1, \dots, z_l\}$

$z_k = x_i y_j$ for some x_i and y_j

Then $E(XY) = \sum_k z_k P\{Z = z_k\}$ $\left(\because \{Z = z_k\} = \bigsqcup_{i,j : x_i y_j = z_k} \{X = x_i, Y = y_j\}\right)$

$$= \sum_k \sum_{i,j : x_i y_j = z_k} x_i y_j P\{X = x_i, Y = y_j\}$$

$$= \sum_{i,j} x_i y_j P\{X = x_i\} P\{Y = y_j\} = E(X)E(Y)$$

$\square$